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A puzzle formula for $H^*_{T\times\mathbb{C}^\times}(T^*\mathbb{P}^n)$

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Abstract. We will begin with the work of Davesh Maulik and Andrei Okounkov where they define a "stable basis" for the T-equivariant cohomology ring $H^*_{T\times\mathbb{C}^\times}(T^*Gr_k(\mathbb{C}^n))$, of the cotangent bundle to a Grassmannian. Just as we can compute the product structure of the the cohomology ring of a Grassmannian using Schubert classes as a basis, it is natural to attempt to do the same for the cotangent bundle to a Grassmannian using these Maulik-Okounkov classes as a basis. In this paper I compute the structure constants of both the regular and equivariant cohomology rings of the cotangent bundle to projective space, using Maulik-Okounkov classes as a basis. First I do so directly in Theorem 3.1, and then I put forth a conjectural positive formula, which uses a variant of Knutson-Tao puzzles, in Conjecture 4.2. The proof of the puzzle formula relies on an explicit rational function identity that I have checked through dimension 9.

Keywords: Schubert calculus, cohomology, puzzles, Maulik Okounkov classes

1 Introduction

Let $\binom{[n]}{k}$ denote the set of strings $\lambda = \lambda_1 \ldots \lambda_n$ consisting of k ones and $n-k$ zeros in arbitrary order. Then Schubert classes, S_λ, which are indexed over λ, form a basis over $\mathbb{Z}$ for the cohomology ring $H^*(Gr_k(\mathbb{C}^n))$. The cup product of any two classes, $S_\lambda S_\mu$, is a sum over the basis $\{S_\nu\}$ with integer coefficients:

$$S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu \text{ where } c^\nu_{\lambda\mu} \in \mathbb{Z}.$$

For geometric reasons these structure constants are non-negative [3]. Determining these integer coefficients is the goal of Schubert calculus, and there are many combinatorial rules which compute them. Allen Knutson and Terry Tao put forward a positive formula for the product structure of both regular[5] and equivariant[4] cohomology on a Grassmannian $Gr_k(\mathbb{C}^n)$ using Schubert classes as a basis, by using a combinatorial tool called Knutson-Tao puzzles. The theory of Knutson-Tao puzzles has been modified to encompass other kinds of cohomology theories, including regular [1] and equivariant[7] K-theory, as well as H^*(2-step manifolds)[2] .

In this paper I compute the structure constants of the regular and equivariant cohomology rings of the cotangent bundle to projective space, using Maulik-Okounkov classes as a basis. First I do so directly using the definitions of Maulik-Okounkov classes as seen in Theorem 3.1. Then I put forth a conjectural positive formula which uses a variant of Knutson-Tao puzzles in Conjecture 4.2. We will see how to perform these computations using the new puzzle pieces and labels, and then show how the proof of the conjecture reduces to an explicit rational function identity that has been checked through dimension 9.

2 Maulik-Okounkov classes

Davesh Maulik and Andrei Okounkov defined in [6] a "stable basis" for the T-equivariant cohomology ring, $H^*_{T\times\mathbb{C}^\times}(T^*Gr_k(\mathbb{C}^n))$, of the cotangent bundle to a Grassmannian. Considering $\lambda,\mu \in \binom{[n]}{k}$ as above, they described classes $\widetilde{M}_\lambda \in H^k_{T\times\mathbb{C}^\times}(T^*Gr_k(\mathbb{C}^n))$, which form a basis over $\mathbb{Z}[\hbar, y_1,\ldots,y_n]$ after inverting $\hbar$. These classes have restrictions $\alpha|_\lambda \in H^*_{T\times\mathbb{C}^\times}$ to fixed points $\mathbb{C}^\lambda \in (T^*Gr_k(\mathbb{C}^n))^{T\times\mathbb{C}^\times}$ of the torus action which satisfy

1. $\widetilde{M}_\lambda|_\mu = 0$ for $\mu \not\geq \lambda$
2. $\widetilde{M}_\lambda|_\lambda = \displaystyle\prod_{i\in[1,k],j\in[1,n-k]} \begin{cases} y_i - y_j & (i,j)\in\lambda \\ \hbar-(y_i-y_j) & (i,j)\notin\lambda \end{cases}$
3. $\hbar \Big| \widetilde{M}_\lambda|_\mu$ for $\mu > \lambda$

and they prove that these conditions uniquely determine this basis $\{\widetilde{M}_\lambda\}$. Just as you can relate Schubert classes to each other using divided difference operators, you can relate Maulik-Okounkov classes to each other using a "deformed reflection operator," R_i, in the following way:

$$R_i \cdot \widetilde{M}_\lambda = \widetilde{M}_{r_i\cdot\lambda} \text{ where } R_i = r_i + \hbar\partial_i$$

This means that we can start at any $\widetilde{M}_\lambda$ and use the various R_is to obtain every other class, as opposed to the Schubert case where you must start from the point class.

In addition to the equivariant classes, Maulik and Okounkov also define a "stable basis," $\{M_\lambda\}$, for the regular cohomology ring $H^*_{\mathbb{C}^\times}(T^*Gr_k(\mathbb{C}^n)) \cong H^*(Gr_k(\mathbb{C}^n))[\hbar]$ over $frac(H^*_{\mathbb{C}^\times}) = \mathbb{Q}(\hbar)$ by using a forgetful map $H^*_{T\times\mathbb{C}^\times}(T^*\mathbb{P}^n) \to H^*_{\mathbb{C}^\times}(T^*\mathbb{P}^n)$ which simply sends all of the y_is to 0. We will focus on the equivariant classes in this paper, as all information about the regular classes can be retrieved from the equivariant case.

3 Non-puzzle formulas

First we will establish a way to determine $c^\nu_{\lambda,\mu}$ without puzzles. So far we have only established formulas, both puzzle and not, for the cases where $k=1$ and $k=n-1$. We begin by looking at the regular cohomology when $k=1$, i.e. $H^*_{\mathbb{C}^\times}(T^*Gr_1(\mathbb{C}^n))$. When we expand to equivariant cohomology, we will use $k=n-1$.

Theorem 3.1 (C).

(1) Consider $\lambda,\mu,\nu \in \binom{[n]}{1}$ so that the 1 is in the ith, jth, and kth spots respectively. Here $c^\nu_{\lambda\mu}$, which we will refer to as c^k_{ij}, corresponds to the coefficient for M_ν in $M_\lambda M_\mu$. Then we get

$$c^k_{ij} = \binom{i+j-k-1}{n-1}$$

(2) Consider $\lambda,\mu,\nu \in \binom{[n]}{n-1}$ so that the 0 is in the ith, jth, and kth spots respectively. Here $c^\nu_{\lambda\mu} = c^k_{ij}$ corresponds to the coefficient for $\widetilde{M}_\nu$ in $\widetilde{M}_\lambda\widetilde{M}_\mu$. Then we get

$$c^k_{ij} = \sum_{i,j\le a\le k} \frac{\hbar \prod_{b<i}(y_a-y_b)\prod_{b>i}(\hbar+y_a-y_b)\prod_{b<j}(y_a-y_b)\prod_{b>j}(\hbar+y_a-y_b)\prod_{b>k}(y_a-y_b)}{\prod_{b\neq a}(y_a-y_b)\prod_{b\ge k}(\hbar+y_a-y_b)}$$

Part (1) follows from the class definitions and codimension arguments.

3.1 Stable Bases $\widetilde{M}_\lambda$: The proof of Theorem 3.1 (2)

Since we are in projective space, i.e. $\lambda \in \binom{[n]}{n-1}$, for the rest of this paper we will refer to $\widetilde{M}_\lambda$ as $\widetilde{M}_i$ for ease of noting where the 0 is in our class. Directly applying the work of Maulik, Okounkov, and Su we get the following lemma.

Lemma 3.2. *In the projective case where $\widetilde{M}_i$ is a stable basis element for the equivariant cohomology of the cotangent bundle to the Grassmannians as described by Maulik and Okounkov, we get*

1. $\widetilde{M}_i|_a = 0$ *for* $a<i$
2. $\widetilde{M}_i|_i = \prod_{b\in[1,i)}(y_i-y_b)\prod_{b\in(i,n]}(\hbar+y_i-y_b)$
3. $\widetilde{M}_i = (R_i\cdot\widetilde{M}_{i+1}) = ((r_i+\hbar\partial_i)\cdot\widetilde{M}_{i+1})$

where r_i is reflection across the axis perpendicular to α_i, and $\partial_i = \frac{1}{\alpha_i}(\mathbb{1}-r_i)$ as before. Recall that the $c^k_{i,j}$s we are looking for are determined by the formula

$$\widetilde{M}_i\cdot\widetilde{M}_j = \sum_k c^k_{ij}\widetilde{M}_k$$

as seen in [6] and [8]. This ring has a standard inner product

$$\langle \alpha, \beta \rangle = \int_{T^*\mathbb{P}^n} \alpha\beta$$

defined by formally applying the Atiyah-Bott/Berline-Vergne equivariant integration formula. We can use the inner product in the following way to define a "dual basis" $\{\widetilde{M}_i^*\} \in H^*_{T\times\mathbb{C}^\times}(T^*\mathbb{P}^{n-1}) \otimes_{H^*_{T\times\mathbb{C}^\times}} frac(H^*_{T\times\mathbb{C}^\times}(T^*\mathbb{P}^{n-1}))$:

$$\langle \widetilde{M}_i, \widetilde{M}_j^* \rangle = \delta_{ij}$$

Here $R_i' = r_i - \hbar\partial_i$ relates the dual classes in the following way:

$$\widetilde{M}_{i+1}^* = (R_i' \cdot \widetilde{M}_i^*) = ((r_i - \hbar\partial_i) \cdot \widetilde{M}_i^*)$$

Note that elements of this dual basis are *not* cohomology classes and do not have a geometric interpretation. We can use R_i and R_i' on the stable basis and the dual basis respectively to get a complete description of the restrictions as

Lemma 3.3. *In the projective case where $\widetilde{M}_i$ is a stable basis element for the equivariant cohomology of the cotangent bundle to the Grassmannians as described by Maulik and Okounkov, we get*

$$\widetilde{M}_i|_a = \prod_{b\in[1,i)} (y_a - y_b) \prod_{b\in(i,n]} (\hbar + y_a - y_b)$$

$$\widetilde{M}_i^*|_a = \prod_{b\in[1,i)} (\hbar + y_a - y_b) \prod_{b\in(i,n]} (y_a - y_b)$$

Proof. This clearly holds for $a = i$. Lemma 3.2 gives us a complete description of $\widetilde{M_n}$ since we know $\widetilde{M}_n|_i = 0$ for all $i \neq n$. Thus using induction and part (3) of Lemma 3.2 we get that if $a > i+1$:

$$\begin{aligned}
(R_i \cdot \widetilde{M}_{i+1})|_a &= \tfrac{\hbar}{y_{i+1}-y_i} \prod_{b<i+1} (y_a - y_b) \prod_{b>i+1} (\hbar + y_a - y_b) + \\
&\quad \tfrac{y_{i+1}-y_i-\hbar}{y_{i+1}-y_i} \prod_{b<i+1} (y_a - y_{r_i\cdot b}) \prod_{b>i+1} (\hbar + y_a - y_{r_i\cdot b}) \\
&= \prod_{b<i} (y_a - y_b) \prod_{b>i+1} (\hbar + y_a - y_b) [\tfrac{\hbar(y_a-y_i)+(y_{i+1}-y_i-\hbar)(y_a-y_{i+1})}{y_{i+1}-y_i}] \\
&= \prod_{b<i} (y_a - y_b) \prod_{b>i+1} (\hbar + y_a - y_b) [\hbar + y_a - y_{i+1}] \\
&= \widetilde{M}_i|_a
\end{aligned}$$

which is what we want. We can use similar arguments to get that $(R_i \cdot \widetilde{M}_{i+1})|_{i+1} = \widetilde{M}_i|_{i+1}$ and $(R_i \cdot \widetilde{M}_{i+i})|_i = \widetilde{M}_i|_i$. Now that we have proved the first statement we can use the fact that

$$\langle \widetilde{M}_i, \widetilde{M}_j^* \rangle = \delta_{ij}$$

to show that

$$\widetilde{M}_1^*|_1 = \prod_{b>1}(y_1 - y_b)$$

The rest follows from an inductive argument. □

Once we have all of these definitions, the proof of part (2) of our theorem becomes a simple computation using equivariant localization:

Proof. Plugging in the above definitions we get that

$$\begin{aligned}
c_{ij}^k &= \int_{T^*\mathbb{P}^n} \widetilde{M}_i\widetilde{M}_j\widetilde{M}_k^* \\
&= \sum_{a\in[max(i,j),k]} \frac{\widetilde{M}_i|_a\cdot\widetilde{M}_j|_a\cdot\widetilde{M}_k^*|_a}{\prod_{b\neq a}(y_a-y_b)(\hbar+y_a-y_b)} \\
&= \sum_{a\in[max(i,j),k]} \frac{\prod_{b<i}(y_a-y_b)\cdot\prod_{b>i}(\hbar+y_a-y_b)\cdot\prod_{b<j}(y_a-y_b)\cdot\prod_{b>j}(\hbar+y_a-y_b)\cdot\hbar\cdot\prod_{b>k}(y_a-y_b)}{\prod_{b\neq a}(y_a-y_b)\cdot\prod_{b\geq k}(\hbar+y_a-y_b)}
\end{aligned}$$

□

While this is a complete description of the structure coefficients, it is an unsatisfying one, given that we know the coefficients to be polynomials which are a positive linear combination of $(y_a - y_b)$ and $(\hbar - (y_a - y_b))$ where $a > b$. The above formula is not only not positive, since there are no restrictions on a and b, but it is not even obviously a polynomial. The puzzle formula will provide us with a much more efficient and satisfying way of calculating these coefficients.

4 Puzzle formula in $H^*_{T\times\mathbb{C}^\times}(T^*Gr_{n-1}(\mathbb{C}^n))$

4.1 Puzzle approach

In previous uses of Knutson-Tao puzzles we see puzzle pieces of varying shapes with specific, usually integer, boundary labels. In this case all puzzle pieces will be unit **nablas** (triangles) or **deltas** (upside down triangles). In order to preserve the number of 1s and 0s on the boundary of our puzzles, each puzzle piece will have side labels read clockwise (a, b, c) which are linear combinations of 1, ω, and ω^2 where $\omega^3 = 1$ and which satisfy $a + b\omega + c\omega^2 = 0$. While these labels and boundary conditions are irrelevant to the statement of the puzzle conjecture, they were instrumental in the process of finding the puzzle pieces, so I include them here to show where the side labels are coming from.

We will consider a **fiefdom** in a puzzle to be a smallest collection of puzzle pieces so that the boundary of the fiefdom is all 1s and 0s. Fiefdoms will become particularly important as we assign weights to equivariant puzzles.

In T-equivariant cohomology $c^{\nu}_{\lambda,\mu} \in \mathbb{Z}[\hbar, y_1, \ldots, y_n]$, so we want to assign to each puzzle a **weight** which lives in $\mathbb{Z}[\hbar, y_1, \ldots, y_n]$, so that when we take the sum over puzzles, we get the $c^{\nu}_{\lambda,\mu}$ that we have described in the previous section.

4.2 Puzzle pieces and the puzzles they create

Using Sage, I was experimentally determined for $n \leq 9$ that we get the correct puzzles by filling in each $\Delta^{\nu}_{\lambda,\mu}$-puzzle with following pieces:

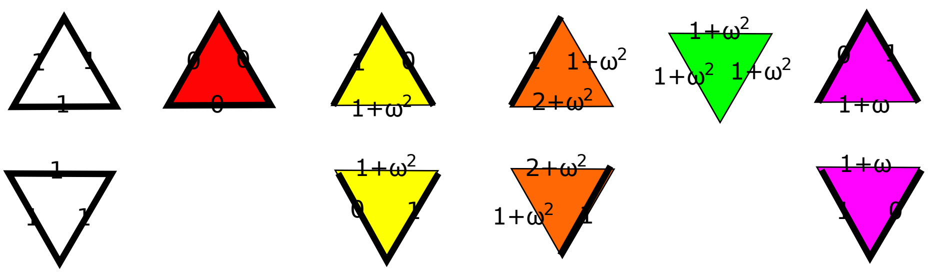

Figure 1: Colored Maulik-Okounkov puzzle pieces.

Note that the equivariant pieces on the right can only be oriented as shown with the $1 + \omega$ label appearing on the horizontal puzzle piece boundary, while the other pieces can be rotated at will. The coloring the of puzzle pieces and bolding the edges of the fiefdoms provides clarity to the diagrams. Once our puzzles have been filled with these pieces we will look at the fiefdoms in the puzzle and assign a weight to each fiefdom, and the product of the weights of the fiefdoms will give us the weight of the puzzle. Using these pieces we always get puzzles that have the shape seen in Figures 2 and 3.

Proposition 4.1. *Each equivariant projective puzzle will have exactly one of the* $(0, 0, 0)$ *delta pieces or exactly one of the* $(1 + \omega^2, 1 + \omega^2, 1 + \omega^2)$ *nabla pieces, which we will call the central piece. The central piece will have three tendrils coming out of it determining how the* 0*s travel to each boundary. The rest of the puzzle will be filled with* $(1, 1, 1)$ *pieces.*

4.3 Weight of an Equivariant puzzle

We can now define what the weight of each fiefdom in the puzzle will be, which we will then multiply together to determine the weight of the whole puzzle.

First consider the NW tendril. By observation we can tell that the only possible way for a 0 to travel from the NW boundary to the central piece is by the equivariant piece or fiefdoms of the following shape:

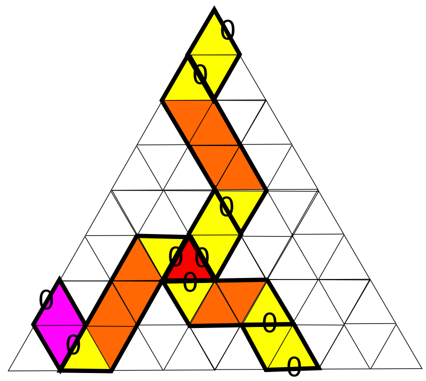

Figure 2: Equivariant Maulik-Okounkov puzzle in $H^*_{T\times\mathbb{C}^\times}(T^*Gr_{n-1}(\mathbb{C}^n)$ with 0 labeled central piece.

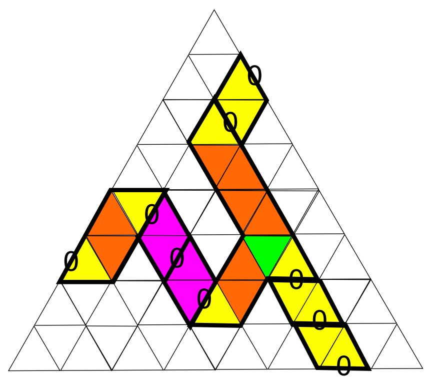

Figure 3: Equivariant Maulik-Okounkov puzzle in $H^*_{T\times\mathbb{C}^\times}(T^*Gr_{n-1}(\mathbb{C}^n)$ with $1+\omega^2$ labeled central piece.

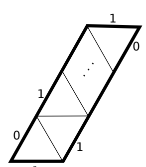

Figure 4: Fiefdom in the NW tendril.

The equivariant pieces will contribute the usual weight, i.e. $y_a - y_b$ where a and b are determined by where SE and SW oriented lines coming from the center of the piece hit the bottom boundary respectively. The other fiefdoms in the tendril contribute $\hbar$.

Looking at the NE tendril we see that there is only one possible fiefdom shape, which is pictured below. However, these fiefdoms contribute different weights based on their size. Fiefdoms that are vertically oriented rhombi with clockwise labels $(1,0,1,0)$ will contribute $\hbar - (y_a - y_b)$ where a and b are determined in the same manner as above. All larger fiefdoms will contribute $\hbar$.

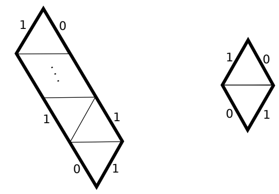

Figure 5: Fiefdoms in the NE tendril.

Looking at the S tendril we see that, again, fiefdoms in this tendril can only have one basic shape. Most of the time these fiefdoms will contribute weight $\hbar + y_a - y_b$ where a and b are determined by where the SE and SW oriented lines originating from the 0 on the northern edge of the fiefdom hot the bottom boundary. However there is a special case if the piece above a fiefdom is the $(0, 0, 0)$ delta piece: then that fiefdom only contributes $\hbar$.

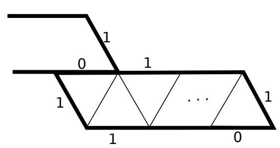

Figure 6: General fiefdom in S tendril.

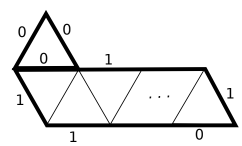

Figure 7: Fiefdom below 0 delta.

Now consider the center fiefdom itself. If it is the $(0, 0, 0)$ delta piece, then it has weight 1. If the center piece is the $(1 + \omega^2, 1 + \omega^2, 1 + \omega^2)$ nabla then the fiefdom it lives in must have the shape seen in Figure 8, and will be assigned weight $\hbar$.

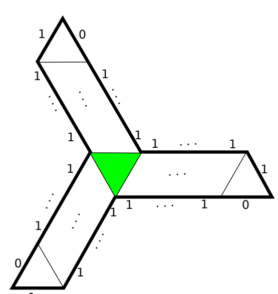

Figure 8: General fiefdom with $1 + \omega^2$ nabla central piece.

Lastly we note that all $(1, 1, 1)$ deltas and nablas have weight 1. Once each fiefdom has been assigned a weight, we get the weight of the whole puzzle by multiplying together the weights of each fiefdom. If the central piece is a 0 piece and it is a rows over from the left and b rows up from the bottom, then there must be a non-trivially weighted fiefdoms in the NW tendril and b non-trivially weighted fiefdoms in the S tendril. Here a and b determine a well-defined spot for the central piece, which forces there to be $n-b-a-1$ non-trivially weighted fiefdoms in the NE tendril. Thus a puzzle of side length n will have an $n-1$ degree homogeneous polynomial as its weight. A similar argument works for puzzles with a $1+\omega^2$ central piece.

We can now return to Figures 2 and 3 and see that they have weights

$$\hbar^3(y_2-y_1)(\hbar-(y_8-y_1))(\hbar-(y_6-y_3))(\hbar+y_6-y_5)$$

and

$$\hbar^2(y_5-y_2)(y_5-y_3)(\hbar-(y_8-y_2))(\hbar+y_7-y_6)(\hbar+y_7-y_5)$$

respectively.

4.3.1 Non-equivariant case

While we already have a nice way to compute our binomial $c^k_{i,j}$ in $H^*_{\mathbb{C}^\times}(T^*Gr_1(\mathbb{C}^n))$, we can also get these coefficients from counting the tilings of a $\Delta^k_{i,j}$-puzzle using the above puzzle pieces minus the equivariant pieces. Since the equivariant fiefdom is the only non-trivially weighted fiefdom which does not contain an $\hbar$ in its weight, then we can also get these coefficients by using the $H^*_{T\times\mathbb{C}^\times}(T^*Gr_{n-1}(\mathbb{C}^n)$ pieces and setting $y_i=0$ for all i, which makes any puzzle with an equivariant fiefdom (weighted y_a-y_b) have weight zero. Thus the binomial coefficient in $H^*_{\mathbb{C}^\times}(T^*Gr_1(\mathbb{C}^n))$ corresponds to the coefficient for $\hbar^{n-1}$ in the proposed puzzle formula for $H^*_{T\times\mathbb{C}^\times}(T^*Gr_{n-1}(\mathbb{C}^n)$. Therefore, once we have a proof for the equivariant puzzle formula, we will have one for the non-equivariant case as well.

4.4 Framework for the proof of the puzzle formula

Define $p^k_{i,j}(\ell,n)$ as the sum of the weights corresponding to size n puzzles with 0s on the boundary at i, j and k with at least ℓ copies of the $(1,0,1,0)$ sideways rhombus stacked at the bottom of the southern tendril as seen in Figure 9. We can use this definition to write down our conjecture that puzzles give us the $c^k_{i,j}$.

Conjecture 4.2. Using the above definition $c^k_{i,j}=p^k_{i,j}(0,n)$, i.e. the total weight of all puzzles with the right boundary.

Using induction we see that the meat of the proof will be in the case of $c^n_{i,1}$.

Lemma 4.3. *(C) The n-dimensional equivariant projective puzzles defined above can be used to compute* $c^k_{i,j}$ *if and only if the same puzzles can be used to compute* $c^n_{i,1}$.

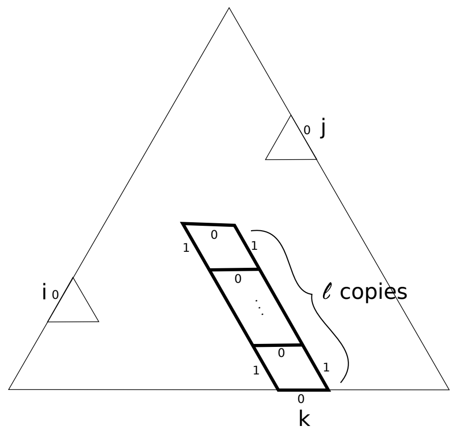

Figure 9: Illustration of $p^k_{i,j}(\ell, n)$.

Then we will only need to show Conjecture 4.2 holds for $p^n_{i,j}(\ell, n)$. Note that the reason we have introduced the notation of $p^k_{i,j}(\ell, n)$ is that it allows us to relate our general puzzle weights $p^n_{i,j}(0, n)$ to weights of smaller puzzles and to $p^k_{i,j}(\ell, n)$ for larger ℓ where $k = n$ or $k = n - 1$.

Theorem 4.4. *The puzzle weight summations $p^k_{i,j}(\ell, n)$ satisfy the recurrence relations*

(1) For $j < n - 1$

$$p^n_{i,j}(1, n) = p^n_{i,j}(0, n) - p^{n-1}_{i,j}(0, n)$$

(2) For $\ell > 1$, and $i < n$

$$p^n_{i,1}(\ell, n) = p^n_{i,1}(\ell - 1, n) - \prod_{b \in [1, \ell-1]} \frac{\hbar + y_n - y_{n-b}}{\hbar + y_{n-1} - y_{n-b-1}} \cdot A$$

where

$$\begin{aligned} A &= (\hbar + y_1 - y_n) p^{n-1}_{i,1}(\ell - 1, n - 1) + (\hbar + y_{n-1} - y_{n-\ell}) p^{n-1}_{i,n-\ell}(\ell - 2, n - 1) \\ &\quad + \hbar \cdot \sum_{a \in [2, n-\ell-1]} p^{n-1}_{i,a}(\ell - 1, n - 1) \end{aligned}$$

The value of increasing ℓ is that once $\ell = n - 2$ there is only one puzzle tiling, illustrated below, and therefore $p^k_{i,1}(n - 2, n) = \hbar \cdot \prod_{b \in [1, n-2]} (\hbar + y_n - y_{n-b})$.

Note that we are only defining our recurrence relation for $j = 1$, and when we reach some $p^n_{i,j}(\ell, n)$ where $j > 1$ we can use induction to get it back down to $p^n_{i,1}(\ell, n)$. We can also use these ideas and recurrence relations to give a parallel definition using the rational function formula as seen in Theorem 3.1:

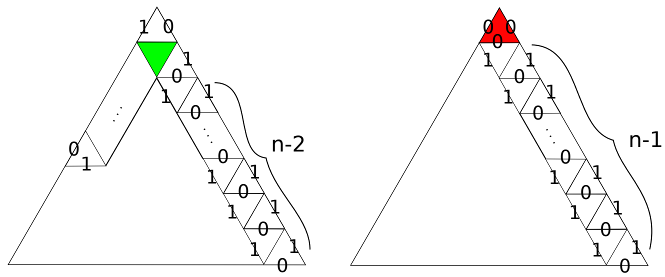

Figure 10: Illustration of $p^k_{i,j}(n-2,n)$ with weight $\hbar \cdot \prod_{b\in[1,n-2]} (\hbar + y_n - y_{n-b})$ for $i < n$ and $i = n$ respectively.

Definition 4.5. *Let*

$$r^n_{i,j}(0,n) := c^n_{i,j}$$

as given by the rational function formula seen in Theorem 3.1. Then we can define $r^n_{i,j}(\ell,n)$ for any $\ell < n-j$ by using the recurrence relations on $p^k_{i,j}(\ell,n)$ in Theorem 4.4.

Once we have these definitions we get the following conjecture.

Conjecture 4.6. Given the above definitions, $r^n_{i,j}(n-2,n) = p^n_{i,j}(n-2,n)$.

Now the $r^n_{i,j}(\ell,n)$s and the $p^n_{i,j}(\ell,n)$s are created using the same recurrence relations and Conjecture 4.2 theorizes that the base cases are also the same, so if Conjecture 4.2 is true, then Conjecture 4.6 will also be true. Similar logic can be used to go in the other direction and then we get the following theorem:

Theorem 4.7. *Conjecture 4.2 is true if and only if Conjecture 4.6 is also true.*

Therefore proving Conjecture 4.6 will give us a proof for our puzzle formula. Conjecture 4.6 has been checked by computer for up to $n = 9$.

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References

[1] A. Buch. "A Littlewood-Richardson rule for the K-theory of Grassmannians". 2000. arXiv:math/0004137.

[2] A. Buch, A. Kresch, K. Purbhoo, and H. Tamvakis. “The puzzle conjecture for the cohomology of two-step flag manifolds”. 2014. arXiv:1401.1725.

[3] S. L. Kleiman and D. Laksov. “Schubert Calculus”. *Amer. Math. Monthly* **79** (1972), pp. 1061–1082.

[4] A. Knutson and T. Tao. “Puzzles and (equivariant) cohomology of Grassmannians”. *Duke Math. J.* **119** (2003), pp. 221–260. DOI.

[5] A. Knutson, T. Tao, and C. Woodward. “The honeycomb model of $GL_n(\mathbb{C})$ tensor products II: Puzzles determine facets of the Littlewood-Richardson cone”. *J. Amer. Math. Soc.* **17** (2004), pp. 19–48. DOI.

[6] D. Maulik and A. Okounkov. “Quantum Groups and Quantum Cohomology”. 2012. arXiv:1211.1287.

[7] O. Pechenik and A. Yong. “Equivariant K-theory of Grassmannians II: The Knutson-Vakil conjecture”. 2015. arXiv:1508.00446.

[8] C. Su. “Restriction Formula for Stable Basis of the Springer Resolution”. 2015. arXiv:1501.04214.